AD-A198 083 TESTS OF FIT USING SAMPLE MOMENTS(U) STANFORD UNIV CA DEPT OF STATISTICS I D CURRIE ET AL. 23 JUN 88 TR-405 N00014-86-K-0156 1/1

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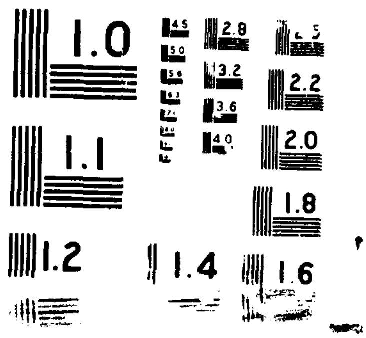
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TESTS OF FIT USING SAMPLE MOMENTS

BY

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TECHNICAL REPORT NO. 405

JUNE 23, 1988

Prepared Under Contract

N00014-86-K-0156 (NR-042-267)

For the Office of Naval Research

Herbert Solomon, Project Director

DEPARTMENT OF STATISTICS

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1. INTRODUCTION

Under regularity conditions, the infinite set of moments (when these exist) characterise a distribution; thus it is appealing to use the first s sample moments, or functions of these, as test statistics for goodness of fit. For example, the statistics b_1 and b_2 have long been used to test for normality, using the first four sample moments. Gurland and Dahiya (1970) and Dahiya and Gurland (1972), in articles hereafter referred to as GD1 and GD2, developed this approach in a systematic way by comparing functions of sample moments to the corresponding functions of population moments; these functions were chosen to be linear in any unknown parameters of the distributions.

In this article we explore some theoretical aspects of the Gurland-Dahiya method, and also give some details of the techniques when applied to tests for the normal, exponential, and Gamma distributions. The basic test statistic of Gurland and Dahiya, $\hat{Q}_t$ below, has an asymptotic χ_t^2 distribution with $t=s-q$, when q parameters must be estimated and s sample moments are used. We show how $\hat{Q}_t$ can be decomposed into t components $\hat{C}_j$ each with an asymptotic χ_1^2 distribution, and such that

$\hat{Q}_t = \hat{Q}_{t-1} + \hat{C}_t$. Thus as a new sample moment is added, the new test statistic contains the previous statistic plus a new asymptotically independent term.

The statistic $\hat{Q}_t$ thus breaks down into components in a manner similar to the Neyman (1937) statistic, or to EDF statistics (Durbin and Knott, (1972) Stephens (1974)); each new component might be expected to test for a new departure from the tested distribution. When applied to a test of normality, with $s = 3$ or 4, the statistics b_1 and b_2 result naturally from the method.

Also, we show that the test statistic for exponentiality, using $s = 2$ moments with only the scale parameter unknown (so $q = 1$) is equivalent to several other statistics already proposed by various authors, often from quite different points of view.

2. THE TEST STATISTICS

2.1 As far as is practicable, we use the notation of GD1 GD2. The symbol ', in the matrix algebra below, will denote the transpose of a column vector or of a matrix. In standard notation, the same symbol will be used to refer to population moments or sample moments (μ'_j and m'_j respectively) about the origin. In context, there should be no confusion.

2.2 Suppose the null hypothesis is

> H_0: a random sample $X_1, X_2, \ldots, X_n$ comes from the continuous density $f(x;\theta)$,

where $\theta' = (\theta_1, \ldots, \theta_q)$ is a row vector of q unknown parameters. We assume for simplicity that all parameters in the distribution are unknown, but the treatment may easily be modified when only a subset of parameters in $f(x;\theta)$ is unknown.

2.3 Suppose ζ_i, $i = 1,\ldots,s$ are functions of the population moments μ_j', $j = 1,\ldots,s$, chosen so that ζ_i is linear in $\theta_1,\ldots,\theta_q$. Thus, writing $\zeta' = (\zeta_1,\ldots,\zeta_s)$, we have $\zeta = W\theta$ where W is an $s \times q$ matrix of known constants.

2.4 Let h_i be the same vector as ζ_i but with sample moments $m_i' = \Sigma_{r=1}^{n} (X_r)^i/n$ replacing the population moments μ_i', and let $h' = (h_1,\ldots,h_s)$; h is a consistent estimator of ζ.

2.5 Consider the vector statistic $\sqrt{n}\,(h-\zeta)$, which measures the difference between h and ζ. Its covariance matrix is $\Sigma = JGJ'$, where G is the $s \times s$ symmetric matrix with entries $G_{ij} = \mu_{i+j}' - \mu_i'\mu_j'$, and J is the $s \times s$ Jacobian matrix with entries $J_{ij} = (\delta\zeta_i/\delta\mu_j')$, $(i,j=1,\ldots,s)$. Gurland and Dahiya show that, if the (2s)-th moment of $f(x;\theta)$ exists, $\sqrt{n}\,(h-\zeta)$ is asymptotically normal with mean = 0 and covariance matrix Σ; then the asymptotic distribution of

$$(1) \qquad Q = n(h-\zeta)'\,\Sigma^{-1}\,(h-\zeta)$$

is χ_s^2.

Further, if Σ is replaced by $\hat{\Sigma}$, a consistent estimator of Σ, the asymptotic distribution of

$$(2) \qquad Q^* = n\,(h-\zeta)'\,\hat{\Sigma}^{-1}\,(h-\zeta)$$

is also χ_s^2. In general, Σ contains some of the unknown parameters in θ; $\hat{\Sigma}$ is obtained by replacing θ in Σ by $\tilde{\theta}$, say, where $\tilde{\theta}$ is a consistent estimator of θ.

2.6 Statistics Q and Q^* will be large when the sample vector h is far from ζ, but, as they stand, they cannot be used for testing the fit of the sample to $f(x;\theta)$ because they contain the unknown parameters in θ; in Q these arise in ζ and Σ. However, when $\tilde{\theta}$ is used to replace θ in Σ, the resulting matrix $\hat{\Sigma}$ will no longer be a function of θ; thus θ enters Q^* only through ζ. Gurland and Dahiya propose that $\hat{Q}_t$, the minimum value of Q^* as θ varies, can be used as a goodness of fit statistic; furthermore, if the minimum occurs for $\theta = \hat{\theta}$, then $\hat{\theta}$ is an estimate of θ, analogous to other "minimum chi square" estimators. The values of $\hat{\theta}$ and $\hat{Q}_t$ are found as follows. Define matrices

$$
\begin{aligned}
&Z = W'\Sigma^{-1}W \quad \text{and} \quad \hat{Z} = W'\hat{\Sigma}^{-1}W \\
(3) \qquad &R = WZ^{-1}W'\Sigma^{-1} \quad \text{and} \quad \hat{R} = W\hat{Z}^{-1}W'\hat{\Sigma}^{-1} \\
&A = \Sigma^{-1}(I_s-R) \quad \text{and} \quad \hat{A} = \hat{\Sigma}^{-1}(I_s-\hat{R})
\end{aligned}
$$

where $\hat{\Sigma}$ is obtained as described above, and where I_s is the $s \times s$ identity matrix. Note that Z and $\hat{Z}$ are $q \times q$; R, $\hat{R}$, A and $\hat{A}$ are $s \times s$ and R and $\hat{R}$ are idempotent. Then (GD1, GD2)

$$
\begin{aligned}
&\hat{\theta} = \hat{Z}^{-1}W'\hat{\Sigma}^{-1}h, \quad \text{and} \\
(4) \qquad &\hat{Q}_t = nh'\hat{A}h. \quad \text{Define also} \\
&Q_t = nh'Ah.
\end{aligned}
$$

Note that, to calculate $\hat{Q}_t$, it is not required to know $\hat{\theta}$.

2.7 <u>Notation</u> In the following sections we shall examine the properties of Q_t and $\hat{Q}_t$. New matrices and vectors will be defined as required; for convenience, the most important of these can be listed together as follows:

matrix T: $\Sigma = TT'$ where T is $s \times s$ lower triangular (definition of T).
matrix K: $Z = KK'$ where K is $q \times q$ lower triangular (definition of K).
matrix L: $L = T^{-1}WZ^{-1}W'(T')^{-1}$, a symmetric and idempotent $s \times s$ matrix.
$S = I_s - L$, a symmetric and idempotent $s \times s$ matrix.

P is the matrix which diagonalises L: thus $L = P\Delta_1 P'$. The leading sub matrix of Δ_1 of order q is I_q and other entries of Δ_1 are zero.
Vector g of length s is $g = T^{-1}h$, with components $g_1, g_2, \ldots, g_s$.
Vector ξ of length s is $\xi = P'g = P'T^{-1}h$, with components $\xi_1, \xi_2, \ldots, \xi_s$.
When $\hat{\Sigma}$ is used instead of Σ the matrices become $\hat{T}$, $\hat{K}$, $\hat{L}$, $\hat{S}$, $\hat{P}$ and the vectors become $\hat{g}$ and $\hat{\xi}$.

3. THE STATISTIC $\hat{Q}_t$

3.1 Gurland and Dahiya (GD1) showed that, on H_0, the distributions of Q_t and the statistic $\hat{Q}_t$ were each asymptotically χ^2_t, with $t = s - q$. We now extend these results to show how the statistics can be decomposed into components.

<u>Theorem 1.</u> <u>There exists a set of components</u> C_i, $i = 1,2,\ldots,$ <u>such that</u> $Q_t = \Sigma_{i=1}^t C_i$, <u>where</u> $t = s - q$; <u>asymptotically</u>, the C_i <u>are independent</u>, <u>each with a</u> χ^2_1 <u>distribution</u>. <u>There also exists a set</u> $\hat{C}_i$, <u>such that</u> $\hat{Q}_t = \Sigma_{i=1}^t \hat{C}_i$; <u>asymptotically the</u> $\hat{C}_i$ <u>are indeoendent each with a</u> χ^2_1 <u>distribution</u>.

Thus $Q_{t+1} = Q_t + C_{t+1}$ and $\hat{Q}_{t+1} = \hat{Q}_t + \hat{C}_{t+1}$; that is, a new asymptotically independent component is added to Q_t to obtain Q_{t+1} , and to $\hat{Q}_t$ to obtain $\hat{Q}_{t+1}$.

For ease of notation, the proof below is given for Q_t only; the proof for $\hat{Q}_t$ is very similar. In what follows, distribution theory will mean asymptotic distribution theory. In this section, when $s+1$ sample moments are used in part 2 of the proof, the vectors and matrices will carry an asterisk, for example, h^*, Σ^*, $\hat{T}^*$.

3.2 Proof of Theorem 1. Part 1

Decomposition of Q_t . Suppose $\Sigma = TT'$ where T is a lower triangular $s \times s$ matrix. Then $Z = W'\Sigma^{-1}W$ may be written $Z = KK'$ where K is a lower triangular $q \times q$ matrix. (Note, however, that K is not $W'(T')^{-1}$) .
Let $h = Tg$, so that $g = T^{-1}h$; then $Q_t = nh'Ah$ becomes

$$Q_t = ng'T^{-1}(I_s-R)Tg \ .$$

Hence

$$Q_t = ng'(I_s-L)g \tag{5}$$

where

$$\begin{aligned} L &= T^{-1}WZ^{-1}W'\Sigma^{-1}T \\ &= T^{-1}WZ^{-1}W'(T')^{-1} \ . \end{aligned}$$

It is easily shown that L is symmetric and idempotent; thus matrix $S = I_s - L$ is also symmetric and indempotent.

<u>Comment</u>. In many situations (e.g. for tests of normality and exponentiality, to be discussed in Sections 4 and 5 below) L and S are also diagonal, with diagonal entries 1 or 0. Define Δ_1 and Δ_2 to be

$$\Delta_1 = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{s-q} \end{bmatrix}$$

where the 0 matrices have the necessary dimensions to make both Δ_1 and Δ_2 of dimensions $s \times s$. Then L will equal Δ_1 and S will equal Δ_2, and, from (5), we have

$$Q_t = ng'Sg . \tag{6}$$

It at once follows that, if $t = s - q$,

$$Q_t = n \sum_{j=1}^{t} C_j , \tag{7}$$

where <u>component</u> $C_j = ng_{q+j}^2$, $j = 1,\ldots,t$. The independence of C_j follows because g_{q+j} is normal $N(0,1)$ and the covariance of g is $T^{-1}\Sigma(T^{-1})' = I$. When $S = I-L$ is not itself diagonal, it may be written $S = P\Delta_2P'$ where P is orthogonal. Then let $\xi = P'g = P'T^{-1}h$ and we have

$$Q_t = n\xi'\Delta_2\xi = n \sum_{i=q+1}^{s} \xi_i^2 . \tag{8}$$

Thus again Q_t is the sum of components $C_j = n\xi_{q+j}^2$, $j = 1,\ldots,t$; each ξ_{g+j} is $N(0,1)$, and the covariance matrix is $P'T^{-1}\Sigma(T^{-1})'P = I$, so that these components are independent and χ_1^2 distributed. Corresponding results hold for $\hat{Q}_t$, which becomes $n\,\Sigma_{i=q+1}^{s}\,\hat{\xi}_i^2$, using the notation of Section 2.7.

Proof of Theorem 1. Part 2

To complete Theorem 1, we must show that the same set of components arises in Q_{t+1} (or $\hat{Q}_{t+1}$) as in Q_t (or $\hat{Q}_t$) with the addition of one new term. For this purpose (pursuing the general case when L and S are not diagonal) we show how matrix P is constructed so that $S = P\Delta_2P'$ or equivalently $L = P\Delta_1P'$.

Construction of matrix P. The $s \times s$ orthogonal matrix P is constructed as follows: let $P_1 = T^{-1}W(K')^{-1}$, an $s \times q$ matrix, and let P_2 be an $s \times (s-q)$ matrix, such that $P_2'P_1 = 0$ and $P_2'P_2 = I$; then construct

$$P = [P_1 : P_2] \ .$$

Lemma 1: P is an orthogonal matrix.

Proof. To show that P is orthogonal we need $P_1'P_1 = I_q$. We have

$$\begin{aligned} P_1'P_1 &= K^{-1}W'(T')^{-1}\,T^{-1}W(K')^{-1} \\ &= K^{-1}W'\Sigma^{-1}\,W(K')^{-1} \\ &= K^{-1}Z(K')^{-1} \\ &= K^{-1}KK'(K')^{-1} \\ &= I_q \quad \text{as required.} \end{aligned}$$

Lemma 2: $LP_2 \equiv 0$.

Proof. Since $P_2'P_1 = 0$ we obtain

$$P_2'T^{-1}W(K')^{-1} = 0 ; \tag{9}$$

thus

$$P_2'T^{-1}W = 0 , \tag{10}$$

and

$$LP_2 = T^{-1}WZ^{-1}W'(T')^{-1}P_2 = T^{-1}WZ^{-1}(P_2'T^{-1}W)' = 0 . \tag{11}$$

Lemma 3: P is the matrix which diagonalises L ; $L = P\Delta_1 P'$.

Proof. We have

$$\begin{aligned} P_1'LP_1 &= K^{-1}W'(T')^{-1}T^{-1}WZ^{-1}W'(T')^{-1}T^{-1}W(K')^{-1} \\ &= K^{-1}W'\Sigma^{-1}WZ^{-1}W'\Sigma^{-1}W(K')^{-1} \\ &= K^{-1}ZZ^{-1}Z(K')^{-1} \\ &= K^{-1}KK'(K')^{-1} \\ &= I_q . \end{aligned}$$

By Lemma 2, $LP_2 \equiv 0$, so

$$P'LP = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} L[P_1 P_2] = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} = \Delta_1 .$$

Thus the idempotent matrix L equals $P\Delta_1 P'$.

To complete the proof a similar procedure is now followed in $s+1$ dimensions. We have $P_1^* = (T^*)^{-1}W^*(K^{*\prime})^{-1}$; and, as above, $P_1^{*\prime}P_1^* = I_q$ and $P_1^{*\prime}L^*P_1^* = I_q$. Since Σ is the leading submatrix of Σ^* , it follows that

$$T^* = \begin{bmatrix} T & 0 \\ t' & c \end{bmatrix}$$

and

$$(T^*)^{-1} = \begin{bmatrix} T^{-1} & 0 \\ v' & d \end{bmatrix} ,$$

where t' , v' are row vectors of length s , 0 is a column vector of length s , and c and d are scalars. Also

$$W^* = \begin{bmatrix} W \\ w' \end{bmatrix}$$

where w' is a row vector of length q . Therefore

$$P_1^* = \begin{bmatrix} T^{-1} & 0 \\ v' & d \end{bmatrix} \begin{bmatrix} W \\ w' \end{bmatrix} (K^{*\prime})^{-1}$$

$$= \begin{bmatrix} T^{-1}W \\ a' \end{bmatrix} (K^{*\prime})^{-1}$$

where $a' = v'W + dw'$. Define matrix V , of dimensions $(s+1)$ by $(s-q)$:

$$V \equiv \begin{bmatrix} P_2 \\ 0' \end{bmatrix}$$

where $0'$ is a row vector of length $s-q$.

<u>Lemma 4</u>. V <u>is orthogonal to</u> P_1^* .

<u>Proof</u>. $V' P_1^* = V' \begin{bmatrix} T^{-1}W \\ a' \end{bmatrix} (K^{*\prime})^{-1} = P_2' T^{-1} W (K^{*\prime})^{-1} = 0$, from (10) .

Therefore, if we define

$$P_2^* = \begin{bmatrix} P_2 : y \\ 0' : r \end{bmatrix} ,$$

where y is a column vector of length s and r is a scalar such that the last column simply completes P_2^* , we have

$$P^* = [P_1^* : P_2^*] \quad .$$

Now

$$Q_t = ng'(I-L)g = ng'(I-P\Delta_1 P')g = n(P'g)'(I-\Delta_1)P'g$$

$$= n \sum_{i=q+1}^{s} \xi_i^2 \text{ where } \xi = P'T^{-1}h ;$$

that is,

$$\begin{bmatrix} \xi_{q+1} \\ \cdot \\ \cdot \\ \cdot \\ \xi_s \end{bmatrix} = P_2' T^{-1} h \quad .$$

Therefore $Q_t = nh'(T')^{-1} P_2 P_2' T^{-1} h$. For Q_{t+1} we have

$$(T^*)^{-1} h^* = \begin{bmatrix} T^{-1} & 0 \\ v' & d \end{bmatrix} \begin{bmatrix} h \\ h_{s+1} \end{bmatrix} = \begin{bmatrix} T^{-1} h \\ v'h + dh_{s+1} \end{bmatrix} \quad .$$

Hence

$$\begin{bmatrix} \xi^*_{q+1} \\ \cdots \\ \xi^*_{s+1} \end{bmatrix} = P_2^{*\prime} (T^*)^{-1} h^* = \begin{bmatrix} P_2' & 0 \\ y' & r \end{bmatrix} \begin{bmatrix} T^{-1} h \\ v'h + dh_{s+1} \end{bmatrix}$$

$$= \begin{bmatrix} P_2' T^{-1} h \\ y' T^{-1} h + r(v'h + dh_{s+1}) \end{bmatrix} \quad .$$

The first s-q components of Q_{t+1} agree with those of Q_t . This result, together with Part 1 completes the proof of Theorem 1 for Q_t .

4. THE TEST FOR NORMALITY

4.1 We illustrate the above decomposition with the test statistic for normality given in GD1. This statistic is constructed as in Section 2 above, with $\zeta' = \{\mu_1', \log \mu_2, \mu_3, \log(\mu_4/3)\}$. Here μ_2, μ_3, μ_4 are moments about the mean, and are, of course, functions of μ_1', μ_2', μ_3' and μ_4' . Then $\zeta = W\theta^*$ where

$$W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \theta^* = \begin{pmatrix} \theta_1 \\ \theta_2^* \end{pmatrix}$$

with $\theta_1 = \mu_1'$, $\theta_2 = \mu_2$, and $\theta_2^* = \log \mu_2$. Vector h is then given by $h' = \{m_1', \log m_2, m_3, \log(m_4/3)\}$, where m_2, m_3, m_4 are sample central moments. Then (GD1, equation 4.10)

$$\Sigma = \begin{pmatrix} \theta_2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 6\theta_2^3 & 0 \\ 0 & 4 & 0 & 32/3 \end{pmatrix}$$

In this example Σ depends on θ_2 but not on θ_1 (the component elements J and G of Σ do depend on θ_1 but θ_1 drops out in the final calculation; GD1 gives the separate matrices J and G as though θ_1 were zero). The decomposition $\Sigma = TT'$ gives

$$T = \begin{pmatrix} \sqrt{\theta_2} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{6\theta_2^3} & 0 \\ 0 & 2\sqrt{2} & 0 & 2\sqrt{(2/3)} \end{pmatrix} \quad \text{and} \quad T^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\theta_2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6\theta_2^3}} & 0 \\ 0 & -\sqrt{3/2} & 0 & \sqrt{3/8} \end{pmatrix}$$

Further, we have

$$Z = W'\Sigma^{-1}W = \begin{pmatrix} \frac{1}{\tilde{\theta}_2} & 0 \\ & \\ 0 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} ,$$

finally $S = T^{-1}(I_s - R)T$ is

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

S is diagonal, so there is no need for further decomposition of S. Matrices $\hat{\Sigma}$, $\hat{T}$, $\hat{T}^{-1}$ are obtained from Σ, T, T^{-1} by replacing θ_2 by its estimate $\tilde{\theta}_2$; then $\hat{g}$ is $\hat{T}^{-1}h$, and with $\tilde{\theta}_2 = m_2$, we have

$$\hat{g}' = [m_1'/m_2^{1/2};\ (\log m_2)/2^{1/2};\ m_3/(6m_2^3)^{1/2};$$
$$- (\log m_2)(1.5)^{1/2} + \{\log(m_4/3)\}(.375)^{1/2}] .$$

The test statistic $\hat{Q}_2$ is

$$\hat{Q}_2 = n\hat{g}_3^2 + n\hat{g}_4^2 = \hat{C}_1 + \hat{C}_2$$

where

$$\hat{C}_1 = \frac{nm_3^2}{6m_2^3} \quad \text{and} \quad \hat{C}_2 = \frac{3n}{8}\left(-2 \log m_2 + \log \frac{m_4}{3}\right)^2 .$$

4.2 <u>Comments</u>. Component $\hat{C}_1$ is equivalent to $b_1 = m_3^2/m_2^3$, and $\hat{C}_2$ is equivalent to $b_2 = m_4/m_2^2$; b_1 and b_2 are the well-known measures of skewness and kurtosis, often proposed for testing normality. An interesting aspect of the method is that it reveals two functions of moments, $\hat{C}_1^{1/2}$ and $\hat{C}_2^{1/2}$, which are asymptotically $N(0,1)$ and independent, and which, when squared, form the components of the overall statistic $\hat{Q}_2$. GD1 gives the final statistic $\hat{Q}_2$ (there called $\hat{Q}$ in equation 4.15) but not the decomposition. The statistic $\hat{\theta}$ in this case becomes $\hat{\theta} = (m_1', \log m_2)'$.

4.3 <u>Extension to any value of s</u>. These results for $s = 4$ may be extended to more general s. For the calculation of Σ set $\theta_1 = 0$; then $\mu_{2r} = \mu_{2r}' = \theta_2^r(2r)!/(2^r r!)$ and odd moments are zero. GD1 gives J in two parts, the Jacobian J_1 of the transformation from central moments to origin moments, and Jacobian J_2 of the transformation from ξ to central moments; $J = J_2 J_1$. *It may be shown that* $(J_1)_{ii} = 1$, $(J_1)_{i1} = -i\mu_i'$, $i > 1$; $(J_1)_{ij} = 0$ otherwise; and $(J_2)_{ii} = 1$ if i is odd, $(J_2)_{ii} = 1/\mu_i$ if i is even, and $(J_2)_{ij} = 0$ otherwise. Let Σ have entries σ_{ij}; $i,j = 1,\ldots,s$: then

$$\sigma_{11} = \theta_2;\ \sigma_{i1} = 0\,,\quad i > 1\,;\quad \sigma_{1j} = 0\,,\quad j > 1$$

$$\sigma_{ij} = \frac{(i+j)!}{i!j!}\ \frac{(\frac{i}{2})!(\frac{j}{2})!}{(\frac{i+j}{2})!} - 1 \qquad i,j \text{ both even}$$

$$\sigma_{ij} = \left(\frac{\theta_2}{2}\right)^{(\frac{i+j}{2})}\left[\frac{(i+j)!}{(\frac{i+j}{2})!} - \frac{2i!j!}{(\frac{i-1}{2})!(\frac{j-1}{2})!}\right] \qquad i,j \text{ both odd}$$

$$\sigma_{ij} = 0 \text{ otherwise}.$$

Then let T be the $s \times s$ matrix with entries t_{ij} defined by

$$t_{11} = \sqrt{\theta_2} \quad , \quad t_{1j} = 0 \;, \quad j > 1 \;, \quad t_{i1} = 0 \;, \quad i > 1$$

$$t_{2m,2n} = \binom{m}{n} \frac{2^{n-1/2}}{\{\binom{2n-1}{n-1}\}^{1/2}} \;, \quad n \le m \;,$$

$$t_{2m+1,2n+1} = \frac{\theta_2^{m+1/2}(2m+1)!}{2^{m-n}(m-n)!} \, \frac{1}{\{(2n+1)!\}^{1/2}} \quad , \quad n \le m$$

$$t_{i,j} = 0 \quad \text{otherwise.}$$

It may be shown that $\Sigma = TT'$, making use of the identities

$$\sum_{n=0}^{j} \frac{2^{2n}(i+j)!}{(2n)!(i-n)!(j-n)!} = \frac{(2i+2j)!}{(2i)!(2j)!}$$

and

$$\sum_{n=0}^{k} \frac{2^{2n+1}(m+k+1)!}{(m-n)!(k-n)!(2n+1)!} = \frac{(2m+2k+2)!}{(2k+1)!(2m+1)!} \;.$$

Let $U \equiv T^{-1}$ and let U have entries u_{ij} . Then

$$u_{11} = \frac{1}{\sqrt{\theta_2}} \quad , \quad u_{1j} = 0 \;, \quad j > 0 \;; \quad u_{i1} = 0 \;, \quad i > 0$$

$$u_{2m,2n} = (-1)^{m+n}\binom{m}{n} \frac{\{\binom{2m-1}{m-1}\}^{1/2}}{2^{m-1/2}} \;, \quad n \le m$$

$$u_{2m+1,2n+1} = \frac{(-1)^{m+n}\{(2m+1)!\}^{1/2}}{(m-n)!\,2^{m-n}\,\theta_2^{n+1/2}(2n+1)!}\ , \quad n \le m$$

$$u_{ij} = 0 \quad \text{otherwise.}$$

This may be checked by direct calculation of UT . Now W is the $s \times 2$ matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 2 \\ 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \quad \text{and column 1 of } T^{-1}W \text{ is } \begin{pmatrix} \frac{1}{\sqrt{\theta_2}} \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

Column 2 of $T^{-1}W$ has zeros in the odd positions and in the even positions the entries are

$$(T^{-1}W)_{2m,2} = \frac{\{\binom{2m-1}{m-1}\}^{1/2}\ (-1)^m \sum_{n=1}^{m} (-1)^n\, n\binom{m}{n}}{2^{m-1/2}}$$

$$= 1/(2^{1/2}) \quad \text{when } m = 1\ , \text{ and } 0 \text{ if } m \neq 1\ .$$

Thus $T^{-1}W$ is the $s \times 2$ matrix

$$T^{-1}W = \begin{bmatrix} \frac{1}{\sqrt{\theta_2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \\ & \vdots \end{bmatrix}$$

and so

$$W'\Sigma^{-1}W = \begin{bmatrix} \frac{1}{\theta_2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

for all s .

Now $R = W(W'\Sigma^{-1}W)^{-1}W'(T')^{-1}T^{-1}$, so

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 2 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \theta_2 & 0 \\ 0 & 2 \end{bmatrix} \left[\begin{matrix} \frac{1}{\sqrt{\theta_2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{matrix} \;\vdots\; \bigcirc \right] \begin{bmatrix} \frac{1}{\sqrt{\theta_2}} & 0 & \vdots & \\ 0 & \frac{1}{\sqrt{2}} & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ --- & & \vdots & --- \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \vdots & \\ 0 & 1 & \vdots & \bigcirc \\ 0 & 0 & \vdots & \\ 0 & 2 & \vdots & \\ 0 & 0 & \vdots & \\ \vdots & \vdots & \vdots & \end{bmatrix} = \left[W \;\vdots\; \bigcirc \right] .$$

We now compute $S = T^{-1}(I-R)T = I - T^{-1}RT$.

Using the above expression for R we find

$$S = I - T^{-1}[W \vdots 0]T$$

$$= I - [T^{-1}W \vdots 0]T$$

$$= I - \left[\begin{array}{cc:c} \frac{1}{\sqrt{\theta_2}} & 0 & \\ 0 & \frac{1}{\sqrt{2}} & \bigcirc \\ \hdashline & & \end{array}\right] \left[\begin{array}{cc:c} \sqrt{\theta_2} & 0 & \bigcirc \\ 0 & \sqrt{2} & \\ \hdashline --- & & --- \end{array}\right] = \left[\begin{array}{cc:c} 0 & 0 & \bigcirc \\ 0 & 0 & \\ \hdashline \multicolumn{2}{c:}{\bigcirc} & I_{s-2} \end{array}\right] .$$

S is again diagonal. Thus if $\hat{g} = \hat{T}^{-1}h$ where $\hat{T}^{-1}$ is T^{-1} with $\tilde{\theta}_2$ replacing θ_2 , we have the general result

$$\hat{Q}_t = n \sum_{i=3}^{s} \hat{g}_i^2$$

Now vector h has components

$$h_1 = m_1'$$

$$h_{2r+1} = m_{2r+1}$$

$$h_{2r} = \log\left(\frac{2^r r!}{(2r!)} m_{2r}\right) .$$

Because of the structure of $\hat{T}^{-1}$, odd $\hat{g}_i$ involve only odd h_i , and even $\hat{g}_i$ involve only even h_i . We have, for r an integer,

$$\hat{g}_{2r} = \sum_{n=1}^{r} (u_{2r,2n})h_{2n}$$

$$= \sum_{n=1}^{r} (-1)^{r+n}\binom{r}{n} \frac{\sqrt{\binom{2r-1}{r-1}}}{2^{r-1/2}} \log\left(\frac{2^n n!}{(2n)!} m_{2n}\right)$$

$$= (-1)^r \frac{\sqrt{\binom{2r-1}{r-1}}}{2^{r-1/2}} \sum_{n=1}^{r} (-1)^n \binom{r}{n} \log\left(\frac{2^n n!}{(2n)!} m_{2n}\right) .$$

In particular,

$$\hat{g}_4 = \frac{1}{2}\sqrt{\frac{3}{2}} \log\left(\frac{m_4}{3m_2^2}\right) , \quad \text{and} \quad \hat{g}_4^2 = \frac{3}{8}\left[\log \frac{b_2}{3}\right]^2 , \quad \text{as above;}$$

also

$$\hat{g}_6 = \frac{\sqrt{5}}{4} \log\left(\frac{9}{5} \frac{m_6 m_2^3}{m_4^3}\right) \quad \text{and} \quad \hat{g}_6^2 = \frac{5}{16}\left[\log\left(\frac{9}{5} \frac{b_4}{b_2^3}\right)\right]^2 ,$$

where $b_2 = m_4/m_2^2$ as before, and $b_4 = m_6/m_2^3$.

The odd $\hat{g}_i$ are, with r an integer,

$$\hat{g}_{2r+1} = \sum_{n=1}^{r} (u_{2r+1,2n+1})h_{2n+1}$$

$$= \sum_{n=1}^{r} (-1)^{r+n} \frac{\sqrt{(2r+1)!}}{(2n+1)!} \frac{m_{2n+1}}{\tilde{\theta}_2^{n+1/2} \quad 2^{r-n}(r-n)!}$$

$$= (-1)^r \frac{\sqrt{(2r+1)!}}{2^r} \sum_{n=1}^{r} (-1)^n \frac{2^n m_{2n+1}}{(2n+1)!\ \tilde{\theta}_2^{n+1/2}(r-n)!} .$$

In particular, using $\tilde{\theta}_2 = m_2$ as before, we have

$$\hat{g}_3 = \frac{1}{\sqrt{6}} \frac{m_3}{m_2^{3/2}} \quad \text{and} \quad \hat{g}_3^2 = \frac{1}{6} b_1^2 , \quad \text{where} \quad b_1 = \frac{m_3}{m_2^{3/2}} ; \quad \text{also}$$

$$\hat{g}_5 = \frac{1}{\sqrt{120}} (b_3 - 10b_1) , \quad \text{where} \quad b_3 = \frac{m_5}{m_2^{5/2}} .$$

Thus

$$\hat{Q}_4 = n\{\frac{1}{6} b_1^2 + \frac{3}{8}(\log \frac{b_2}{3})^2 + \frac{1}{120} (b_3 - 10b_1)^2 + \frac{5}{16} (\log \frac{9b_4}{5b_2^3})^2 \} ,$$

with, on H_0 , a χ_4^2 distribution asymptotically. Extension to higher order $\hat{Q}_t$ is obvious.

5. THE TEST FOR EXPONENTIALITY

5.1 In this section we give the general decomposition of $\hat{Q}_t$ for the test for the exponential distribution $F(x) = 1 - \exp(-x/\theta)$, $x > 0$. Following GD2 we define

$$\zeta' = [\mu_1', \frac{\mu_2'}{\mu_1'}, \frac{\mu_3'}{\mu_2'}, \ldots, \frac{\mu_s'}{\mu_{s-1}'}] ;$$

then $\zeta = W\theta$, where $W' = [1,2,\ldots,s]$.

Vector h is ζ with m_i' replacing μ_i' . The covariance matrix Σ of $\sqrt{n}\, h$ is then known to have entries

$$\sigma_{ij} = \frac{\theta^2 ij(i+j-2)!}{(i-1)!(j-1)!} , \quad i \geq j .$$

We now write $\Sigma = TT'$ where the entries of T are

$$t_{ij} = \theta i\binom{i-1}{j-1}, \quad i \geq j$$

$$= 0, \quad i < j .$$

Then $U = T^{-1}$ has entries

$$u_{ij} = \frac{1}{\theta}\frac{(-1)^{i+j}}{j}\binom{i-1}{j-1}, \quad i \geq j$$

$$= 0, \quad i < j .$$

Further results are: $(T^{-1}W)' = (\frac{1}{\theta}, 0, 0, \ldots, 0)$; $Z = 1/\theta^2$; R has first column entries $1,2,\ldots,s$ and all other entries zero; $T^{-1}R$ has $1/\theta$ in row 1, column 1 and zeros elsewhere. Finally S becomes diagonal, as for the normal case:

$$S = \begin{bmatrix} 0 & 0 \\ 0 & I_{s-1} \end{bmatrix}$$

and further decomposition of S is unnecessary. Let g_i be the i-th component of $\hat{g} = \hat{T}^{-1}h$: using $\tilde{\theta}$ as the estimate of θ in $\hat{T}$, we have, with $m_0' \equiv 1$:

$$\hat{g}_i = \frac{1}{\tilde{\theta}} \sum_{j=1}^{i} \frac{(-1)^{i+j}\binom{i-1}{j-1}}{j} \frac{m_j'}{m'_{j-1}}, \quad i = 1,2,\ldots$$

In particular, writing s_j for m_j' for ease of notation, and using $\tilde{\theta} = s_1$ we have

$$\hat{g}_1 = 1 \ ; \quad \hat{g}_2 = -1 + \frac{s_2}{2s_1^2} \ ;$$

$$\hat{g}_3 = 1 - \frac{s_2}{s_1^2} + \frac{s_3}{3s_1s_2} \ ; \quad \hat{g}_4 = -1 + \frac{3s_2}{2s_1^2} - \frac{s_3}{s_1s_2} + \frac{s_4}{4s_1s_3} \ .$$

Note that $\hat{g}_2$ is equivalent to

$$\frac{s_2}{s_1^2} = n \sum_{r=1}^{n} (X_r)^2 / (\sum_{r=1}^{n} X_r)^2 \ .$$

The test statistic $\hat{Q}_t$ is then $n \Sigma_{i=1}^{t} \hat{g}_{1+i}^2$.

5.2 <u>Comment</u>. <u>Connection with a test for uniformity</u>. Suppose the sample values X_i are placed in sequence on a line, and let $v_{(j)} = \Sigma_{i=1}^{j} X_i$; suppose the $v_{(j)}$ are divided by the sum of the X_i , $z = \Sigma_{i=1}^{n} X_i$, to give values $u_{(j)} = v_{(j)}/z$; it is well-known that, on H_0 , the $u_{(j)}$, $j = 1,\ldots,n-1$ are the order statistics of a sample of size $n-1$ from the uniform distribution on $[0,1]$. The values X_i/z are the <u>spacings</u> d_i between the $u_{(j)}$, and a test for uniformity, proposed by Greenwood, is based on $G = \Sigma_{i=1}^{n} d_i^2$. G is then $s_2/(ns_1^2)$ and so component $\hat{g}_2$ above is equivalent to G . There has recently been a revival of interest in G , and percentage points for finite n have been given by Burrows (1979), Currie (1981) and Stephens (1981). These points show that $\sqrt{n}\, g_2$ converges only very slowly to its asymptotic $N(0,1)$ distribution.

6. <u>TEST FOR THE GAMMA DISTRIBUTION</u>

6.1 The two previous illustrations, tests for normality and exponentiality, were interesting because the general form could be produced for any s , and also because for small s the test components reduced to already well-known test statistics. In DG2 the $\hat{Q}_t$ statistic is also discussed for the Gamma distribution. This situation is important because although many tests exist (EDF tests, for example) for populations in which the unknown parameters are location or scale, far fewer tests are available for the case, as here, where a shape parameter is unknown. Unfortunately it does not appear to be straightforward to give very general results, but we now develop the components of $\hat{Q}_1$ and $\hat{Q}_2$.

The null hypothesis is H_0: the population for X is

$$f(x) = \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \;;\quad x > 0 \;;\quad \alpha,\beta > 0 \;.$$

Thus $q = 2$. In DG2 new parameters are defined: $\theta_1 = \alpha\beta$, $\theta_2 = \beta$ (we have here changed the DG2 notation). Then the vector ζ is given by

$$\zeta' = \left(\kappa_1, \frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \ldots, \frac{\kappa_s}{\kappa_{s-1}}\right) ,$$

where κ_j is the j-th population cumulant $\kappa_j = (j-1)!\alpha\beta^j$. Then if $\theta' = (\theta_1, \theta_2)$ we have $\zeta = W\theta$ with

$$W' = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 2 & & s-1 \end{bmatrix}$$

Vector h is vector ζ with sample cumulants k_i replacing population cumulants κ_i. Considerable algebra gives, for $s = 3$,

$$\Sigma = \frac{\beta^2}{\alpha}\begin{bmatrix} \alpha^2 & \alpha & 2\alpha \\ \alpha & 3+2\alpha & 10+8\alpha \\ 2\alpha & 10+8\alpha & 48+50\alpha+6\alpha^2 \end{bmatrix} .$$

Then

$$T = \frac{\beta}{\sqrt{\alpha}}\begin{bmatrix} \alpha & 0 & 0 \\ 1 & \gamma & 0 \\ 2 & 4\gamma & \delta \end{bmatrix}$$

and

$$T^{-1} = \frac{\sqrt{\alpha}}{\beta}\begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ -\frac{1}{\alpha\gamma} & \frac{1}{\gamma} & 0 \\ \frac{2}{\alpha\delta} & -\frac{4}{\delta} & \frac{1}{\delta} \end{bmatrix}$$

where $\gamma^2 = 2(1+\alpha)$ and $\delta^2 = 6(1+\alpha)(2+\alpha)$. Finally $I_s - R$ is

$$I_s - R = \frac{1}{3\alpha+10}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & -6\alpha-28 & 3\alpha+14 \end{bmatrix}$$

and

$$S = T^{-1}(I_s - R)T = \frac{1}{3\alpha+10}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & \{12(2+\alpha)\}^{1/2} \\ 0 & \{12(2+\alpha)\}^{1/2} & 3(2+\alpha) \end{bmatrix} .$$

S is idempotent of rank 1 as required, but, in contrast to the normal and exponential cases, S is not diagonal. Decomposition into $S = P\Delta P'$ gives

$$P = \frac{1}{\sqrt{3\alpha+10}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \{6+3\alpha\}^{1/2} & 2 \\ 0 & -2 & \{6+\alpha\}^{1/2} \end{bmatrix}$$

and $\Delta_{ij} = 0$ for all i,j except $\Delta_{33} = 1$.

Then using 3.2.8 we have $\xi = P'T^{-1}h$, and $\hat{\xi} = \hat{P}'\hat{T}^{-1}h$, where $\hat{P}$, $\hat{T}$ are derived from P , T by replacing parameters α and β by consistent estimators $\hat{\alpha}$ and $\hat{\beta}$ (GD2 use moment estimators). Because only Δ_{33} is not zero in matrix Δ ,

$$Q_t = n\xi'\Delta\xi = n\xi_3^2 , \quad \text{and similarly} \quad \hat{Q}_t = n\hat{\xi}_3^2 .$$

We need therefore find only $\hat{\xi}_3$. This becomes

$$\hat{\xi}_3 = \frac{1}{\tilde{\beta}}\{\tilde{\alpha}/(10+3\tilde{\alpha})\}^{1/2}w'h$$

where

$$w' = [0, -\frac{2}{(2+2\tilde{\alpha})^{1/2}}, \frac{1}{(2+2\tilde{\alpha})^{1/2}}]$$

and

$$h = [k_1, \frac{k_2}{k_1}, \frac{k_3}{k_2}] .$$

Moment estimators of α and β are $\tilde{\alpha} = \frac{k_1^2}{k_2}$ and $\tilde{\beta} = \frac{k_2}{k_1}$. Thus

$$2+2\tilde{\alpha} = 2(k_2+k_1^2)/k_2 , \quad \text{and} \quad \frac{\tilde{\alpha}}{10+3\tilde{\alpha}} = \frac{k_1^2}{(10k_2+3k_1^2)} ;$$

then

$$\hat{\xi}_3 = \frac{k_1(k_3k_1-2k_2^2)}{\sqrt{2}\, k_2^{3/2}(10k_2+3k_1^2)^{1/2}(k_2+k_1^2)^{1/2}} .$$

and

$$\hat{Q}_1 = \frac{nk_1^2(k_3k_1-2k_2^2)^2}{2k_2^3(10k_2+3k_1^2)(k_2+k_1^2)} .$$

In terms of sample moments this is, writing s_i for m_i' as in Section 5,

$$\hat{Q}_1 = \frac{ns_1^2}{2s_2(s_2-s_1^2)^3(10s_2-7s_1^2)} (s_1s_3 + s_2s_1^2 - 2s_2^2)^2 .$$

Note that $s_1s_3 + s_2s_1^2 - 2s_2^2 = m_1'm_3 - 2m_2^2$, where m_i is the i-th central moment.

6.2 Comment. The calculations above can be considerably simplified by starting with $\zeta' = (\frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \ldots, \frac{\kappa_s}{\kappa_{s-1}})$, that is, by missing out the first component in the above treatment. This has the effect of reducing the matrix and vector dimensions by 1. For $s = 4$ we now pursue this approach.

6.3 Test for Gamma distribution, s=4. Short method. Let

$$\zeta' = \left(\frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \frac{\kappa_4}{\kappa_3}\right) = (\beta, 2\beta, 3\beta) = \beta[1,2,3] .$$

Then $\zeta = W'\beta$ where $W' = [1,2,3]$, and

$$\Sigma = \frac{\beta^2}{\alpha} \begin{bmatrix} 3+2\alpha & 10+8\alpha & 21+18\alpha \\ 10+8\alpha & 48+50\alpha+6\alpha^2 & 132+153\alpha+27\alpha^2 \\ 21+18\alpha & 132+153\alpha+27\alpha^2 & 450 + \frac{1185}{2}\alpha + \frac{315}{2}\alpha^2 + 6\alpha^3 \end{bmatrix}$$

$$= \frac{\beta^2}{\alpha} \begin{bmatrix} a^2 & ab & ad \\ ab & b^2+c^2 & bd+ce \\ ad & bd+ce & d^2+e^2+f^2 \end{bmatrix}$$

where a,b,d,d,e,f are defined by comparing these two expressions for Σ . Decomposition of Σ gives

$$\Sigma = TT' \quad \text{where} \quad T = \frac{\beta}{\sqrt{\alpha}} \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} ;$$

then

$$T^{-1} = \frac{\sqrt{\alpha}}{\beta} \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{b}{ac} & \frac{1}{c} & 0 \\ \frac{be-cd}{acf} & -\frac{e}{cf} & \frac{1}{f} \end{bmatrix}$$

and

$$T^{-1}W = \frac{\sqrt{\alpha}}{\beta}\begin{bmatrix}\frac{1}{a}\\ \frac{2}{c}-\frac{b}{ac}\\ \frac{3}{f}-\frac{2e}{cf}+\frac{be-cd}{acf}\end{bmatrix} = \frac{\sqrt{\alpha}}{\beta}\begin{bmatrix}\frac{1}{a}\\ \frac{2a-b}{ac}\\ \frac{3ac-2ae+be-cd}{acf}\end{bmatrix} = r\ , \quad \text{say}\ .$$

We have $Z = W'\Sigma^{-1}W = W'(T')^{-1}T^{-1}W = r'r$, so that Z is scalar. Matrix L becomes

$$\begin{aligned} L &= T^{-1}RT \\ &= T^{-1}WZ^{-1}W'\Sigma^{-1}T \\ &= Z^{-1}T^{-1}WW'(T')^{-1}T^{-1}T \quad \text{since } Z \text{ is a scalar} \\ &= Z^{-1}T^{-1}WW'(T')^{-1} \\ &= Z^{-1}(T^{-1}W)(T^{-1}W)' = Z^{-1}rr' \end{aligned}$$

Define z to be the normalised vector r , that is, $z = r/(Z)^{1/2}$. Then $L = zz'$.

We now construct P such that $P\Delta P' = L$.

Clearly column 1 of P is vector z . From the solution for $s = 3$ we know column 2 has components $(b-2a,c,0)$, normalized. Lastly column 3 is

$$\begin{bmatrix} -cG \\ (b-2a)G \\ [(b-2a)^2+c^2]f \end{bmatrix} ,$$

normalized, with $G = be-cd-2ae+3ac$. The normalizing factor is m where

$$
\begin{aligned}
m^2 &= c^2G^2+(b-2a)^2G^2 + [(b-2a)^2+c^2]^2 f^2 \\
&= [c^2+(b-2a)^2]G^2 + [c^2+(b-2a)^2]^2 f^2 \\
&= [c^2+(b-2a)^2]\{G^2 + [c^2+(b-2a)^2]f^2\} \quad .
\end{aligned}
$$

Finally $\xi = P'T^{-1}h$. We find ξ , and can then replace α and β by estimates $\tilde{\alpha}$ and $\tilde{\beta}$ to obtain $\hat{\xi}$.

Component ξ_2 is now the same as ξ_3 of Section 6.1; the new ξ_3 is

$$
\xi_3 = \frac{1}{m}\,[-cG, (b-2a)G, \{(b-2a)^2+c^2\}f]\ \frac{\sqrt{\alpha}}{\beta acf} \begin{bmatrix} cf & 0 & 0 \\ -bf & af & 0 \\ be-cd & -ae & ac \end{bmatrix} h
$$

$$
= \frac{1}{m} \cdot \frac{\sqrt{\alpha}}{\beta acf} \begin{bmatrix} -c^2fG - bf(b-2a)G + [(b-2a)^2 + c^2]f(be-cd) \\ af(b-2a)G - aef[(b-2a)^2 + c^2] \\ acf[(b-2a)^2 + c^2] \end{bmatrix}' h
$$

$$
= \frac{1}{m} \cdot \frac{\sqrt{\alpha}}{\beta acf} \begin{bmatrix} acf12(1+\alpha)(3\alpha+8) \\ -acf3(1+\alpha)(26+9\alpha) \\ acf2(1+\alpha)(10+3\alpha) \end{bmatrix}' h
$$

$$= \frac{\sqrt{\alpha}\ (1+\alpha)}{m\beta} \begin{bmatrix} 12(8+3\alpha) \\ -3(26+9\alpha) \\ 2(10+3\alpha) \end{bmatrix}' h$$

with

$$m^2 = [c^2+(b-2a)^2][G^2+\{c^2+(b-2a)^2\}f^2]$$

$$= 2(10+3\alpha)(\alpha+1)12(1+\alpha)^2(2+\alpha)(39+19\alpha+3\alpha^2)$$

$$= 24(1+\alpha)^3(10+3\alpha)(2+\alpha)(39+19\alpha+3\alpha^2) \quad .$$

We obtain

$$\xi_3^2 = \frac{V^2}{24(1+\alpha)(10+3\alpha)(2+\alpha)(39+19\alpha+3\alpha^2)\beta^2}$$

where

$$V = 12(8+3\alpha)\frac{k_2}{k_1} - 3(26+9\alpha)\frac{k_3}{k_2} + 2(10+3\alpha)\frac{k_4}{k_3} \quad .$$

To obtain $\hat{\xi}_3^2$ we replace α and β by their moment estimators $\tilde{\alpha} = k_1^2/k_2$ and $\tilde{\beta} = k_2/k_1$. Then

$\hat{Q}_2 = n(\hat{\xi}_2^2+\hat{\xi}_3^2) = \hat{Q}_1 + \hat{C}_2$ where $\hat{Q}_1$ is given in Section 6.1, and $\hat{C}_2 = n\hat{\xi}_3^2$.

7. FURTHER REMARKS

The techniques which have been explored above lead to goodness-of-fit statistics based on sample moments; the test statistic can be made dependent on more moments by adding new components. These components are asymptotically independent, and can be expected to test for different departures from the tested distribution, analogous to components of EDF statistics (Durbin and Knott, 1972; Stephens, 1974) or those of Neyman's statistic (Miller and Quesenberry, 1978). It has been shown how $\hat{Q}_t$, for small t, leads to tests for normality and exponentiality based on already well-known statistics, and the alternatives for which these are powerful are generally recognized; however, the power of higher-order components to detect important departures must be investigated. A related question is how many components to take, since the addition of too many can weaken the overall power against important alternatives (see Solomon and Stephens 1982 for remarks on Neyman's statistic which are also relevant here; also Durbin and Knott, 1972, Miller and Quesenberry, 1978).

We note also that the technique is not unique: other functions of moments can be made linear in unknown parameters and will lead to other test statistics. Interesting questions then remain on how best to choose ζ. These questions, and the practical questions of producing points for finite n, and power studies, are currently being investigated.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

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Field	Entry
1. REPORT NUMBER	405
2. GOVT ACCESSION NO.	
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle)	Tests Of Fit Using Sample Moments
5. TYPE OF REPORT & PERIOD COVERED	TECHNICAL REPORT
6. PERFORMING ORG. REPORT NUMBER	
7. AUTHOR(s)	Iain D. Currie and Michael A. Stephens
8. CONTRACT OR GRANT NUMBER(s)	N00014-86-K-0156
9. PERFORMING ORGANIZATION NAME AND ADDRESS	Department of Statistics Stanford University Stanford, CA 94305
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS	NR-042-267
11. CONTROLLING OFFICE NAME AND ADDRESS	Office of Naval Research Statistics & Probability Program Code 1111
12. REPORT DATE	June 23, 1988
13. NUMBER OF PAGES	35
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	
15. SECURITY CLASS. (of this report)	UNCLASSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

goodness of fit, moments, normal, exponential, gamma distribution

[illegible]-square [illegible]

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It is shown that the goodness of fit statistic based on moments of Gurland and Dahiya (1970) can be expressed as a sum of components having asymptotically independent χ^2_1 distributions. Expressions are found for the components under the null hypothesis of normality or exponentiality. The first and second components in the normal case are equivalent to the skewness and kurtosis measures b_1 and b_2 respectively. The first component in the exponential case is equivalent to Greenwood's statistic. The gamma distribution is also considered.

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S/N 0102-014-6601

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